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Effect of thermal modulation on the onset of convection in a rotating fluid layer

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Abstract

The stability of a rotating horizontal fluid layer heated from below is examined when, the walls of the layer are subjected to time-periodic temperature modulation. The linear stability analysis is used to study the effect of infinitesimal disturbances. A regular perturbation method based on small amplitude of applied temperature field is used to compute the critical values of Rayleigh number and wavenumber. The shift in critical Rayleigh number is calculated as a function of frequency of modulation, Taylor number and Prandtl number. It is established that the instability can be enhanced by the rotation at low frequency symmetric modulation and with moderate to high frequency lower wall temperature modulation, whereas the stability can be enhanced by the rotation in case of asymmetric modulation. The effect of Taylor number and Prandtl number on the stability of the system is also discussed. We found that by proper tuning of modulation frequency, Taylor number and Prandtl number it is possible to advance or delay the onset of convection.

Keywords: Convection; Thermal modulation; Rotation; Instability

1. Introduction

There has been a growing interest in externally modulated hydrodynamic systems, both theoretically and experimentally. These systems may exhibit novel behavior in response to parametric forcing near a point of instability. Depending on the relative strength and rate of forcing, predictions exist for a variety of responses to the modulation. Among these are the upward or downward shift of the convective threshold compared to the unmodulated problems. There are many works available in the literature, concerning how a time-periodic boundary temperature affects the onset of Rayleigh–Benard convection. Most of the findings related to this problem have been reviewed by Davis [1].

A linear stability analysis of small amplitude temperature modulation is performed by Venezian [2]. He derived the onset criteria using a perturbation expansion in powers of the amplitude of oscillations. He has established that the onset of convection can be delayed or advanced by the out-of or in phase modulation of the boundary temperatures respectively, as compared to the unmodulated system. A problem of low frequency modulation of thermal instability has been investigated by Rosenblat and Herbert [3]. They obtained the asymptotic solution with arbitrary amplitude ratio and made the comparison with known experimental results. Rosenblat and Tanaka [4] have studied the effect of thermal modulation on the onset of Rayleigh–Benard convection. They solved the problem by using the Galerkin technique and discussed the stability using the Floquet theory. It has been found that in general, there is an enhancement of the critical value of the suitably defined Rayleigh number. A weak non-linear stability analysis of thermal modulation has performed by Roppo et al. [5]. They observed that ranges of stable hexagons are produced by the modulation effect near the critical Rayleigh number. These authors have reported that for low frequencies the modulation is destabilizing where as at high frequencies some stabilization is apparent.

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Nomenclature

d	height of the fluid layer
$\mathbf{g}$	gravitational acceleration
l, m	wavenumbers in x, y-directions
p	pressure
Pr	Prandtl number, ν/κ
$\mathbf{q}$	velocity vector, (u, v, w)
Ra	Rayleigh number, $\beta g \Delta T d^3/\nu\kappa$
t	time
T	temperature
Ta	Taylor number, $4d^4\Omega^2/\nu^2$
(x, y, z)	space coordinates
Greek symbols	
α	horizontal wavenumber
β	thermal expansion coefficient
ε	amplitude of modulation
ϕ	phase angle
κ	thermal diffusivity
μ	dynamic viscosity
ν	kinematic viscosity μ/ρ_0
ρ	density
$\mathbf{\Omega}$	angular velocity, $(0, 0, \Omega)$
$\bar{\omega}$	dimensional frequency of modulation
ω	non-dimensional frequency of modulation, $d^2\bar{\omega}/\kappa$
Other symbols	
∇_1^2	horizontal Laplacian operator, $\partial^2/\partial x^2 + \partial^2/\partial y^2$
∇^2	Laplacian operator, $\nabla_1^2 + \partial^2/\partial z^2$
Subscripts/superscripts	
b	basic state
c	critical
0	reference value
*	non-dimensional quantity
′	perturbed quantity

Finucane and Kelly [6] performed both theoretical and experimental investigation of the thermal modulation in a horizontal fluid layer. They found both experimentally and numerically that at low frequencies the modulation is destabilizing, where as at high frequencies it is stabilizing. Thermal convection under external modulation of driving force has been studied by Ahlers et al. [7] using Lorenz model. It is found that in general the modulation stabilizes the conducting state. The non-linear behavior of the model is also studied and shown to reproduce good approximation, most previous theoretical results on modulated convection. Niemela and Donnelly [8] have studied the Rayleigh–Benard convection subject to the external modulation experimentally using Helium-I as working fluid. They observed both positive and negative shift of the convective threshold compared to the unmodulated systems. Recently, Schmitt and Lucke [9], Or and Kelly [10] and Or [11] have also investigated the effect of external modulation on the thermal convection in a horizontal fluid layer.

While the rotation effect on the thermally driven flows is well understood [12,13], there seems to have only one short communication [14], on the effect of rotation on the stability of thermally modulated system. A brief study of the combined effect of thermal modulation and rotation on the onset of convection in a rotating fluid layer was made by Rauscher and Kelly [14] for the case of the lower wall temperature modulation and when Prandtl number equal to unity. They have reported that high Taylor number has a destabilizing effect over a range of frequencies. For small Prandtl numbers, convection in a rotating fluid layer can begin in an oscillatory manner and the modulation might be expected to have more of a resonant effect.

This paper presents the stability analysis of a heated fluid layer subject to both boundary temperature modulation and rotation. We intended to provide a fundamental understanding of how rotation would influence the thermal convection arising from thermal perturbations. As a first attempt, we present a linear stability analysis of a sinusoidally heated fluid system to explore the effect of rotation on oscillating flows. Some of the questions such as how the external field affects the modulated thermally driven convection instability and whether or not the understanding gained from studies on the effect of rotation on unmodulated system is pertinent, remain basically unresolved. We intended to answer these questions by solving the thermally modulated system with rotation using small perturbation technique.

2. Mathematical formulation

We consider a Boussinesq viscous fluid layer confined between two infinite horizontal plates a distance 'd' apart with a vertical downward gravity force acting on it. A Cartesian frame of reference is chosen with the origin in the lower boundary and the z-axis vertically upwards. The fluid layer is subjected to the rotation with an angular velocity $\mathbf{\Omega}$. The axis of rotation is taken along the z-axis. The Boussinesq approximation is applied to account for the effects of density variations. With these assumptions the basic governing equations are

$$\nabla \cdot \mathbf{q} = 0, \tag{1}$$

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla)\mathbf{q} + 2\mathbf{\Omega} \times \mathbf{q} = -\frac{1}{\rho_0}\nabla p + \frac{\rho}{\rho_0}\mathbf{g} + \nu\nabla^2\mathbf{q}, \tag{2}$$

$$\frac{\partial T}{\partial t} + (\mathbf{q} \cdot \nabla)T = \kappa\nabla^2 T, \tag{3}$$

$$\rho = \rho_0[1 - \beta(T - T_0)]. \tag{4}$$

For simplicity, the free-free hydrodynamic boundary conditions are applied at the walls. These conditions are such that the normal velocities are zero and the tangential stresses are zero at both the top and bottom surface. The external driving force is modulated harmonically in time by varying the temperatures of lower and upper horizontal boundary

$$T_0 + \frac{\Delta T}{2}[1 + \varepsilon \cos \bar{\omega} t] \quad \text{at } z = 0, \tag{5}$$

$$T_0 - \frac{\Delta T}{2}[1 - \varepsilon \cos(\bar{\omega} t + \phi)] \quad \text{at } z = d. \tag{6}$$

where ε represents amplitude, $\bar{\omega}$ the frequency and ϕ the phase angle. We consider three types of thermal modulation, viz.,

Case (a): symmetric (in phase, $\phi = 0$),
Case (b): asymmetric (out of phase, $\phi = \pi$),
Case (c): only lower wall temperature modulation ($\phi = -\mathrm{i}\infty$).

2.1. Basic state

Basic state of the fluid is quiescent and is given by

$$\mathbf{q}_\mathrm{b} = 0, \quad T = T_\mathrm{b}(z,t), \quad p = p_\mathrm{b}(z,t), \quad \rho = \rho_\mathrm{b}(z,t). \tag{7}$$

The basic state temperature $T_\mathrm{b}(z,t)$ satisfies the equation

$$\frac{\partial T_\mathrm{b}}{\partial t} = \kappa \frac{\partial^2 T_\mathrm{b}}{\partial z^2}, \tag{8}$$

and the pressure $p_\mathrm{b}(z,t)$ balances the buoyancy force. The solution of Eq. (8) subject boundary conditions (5) and (6) consists of both steady and oscillating parts and is given by

$$T_\mathrm{b} = T_0 + \frac{\Delta T}{2}\left\{\left(1 - \frac{2z}{d}\right) + \varepsilon Re[\{a(\lambda)\mathrm{e}^{\lambda z/d} + a(-\lambda)\mathrm{e}^{-\lambda z/d}\}\mathrm{e}^{-\mathrm{i}\bar{\omega} t}]\right\}, \tag{9}$$

where $\lambda = (1-i)\left(\frac{\bar{\omega} d^2}{2\kappa}\right)^{1/2}$, $a(\lambda) = \frac{\mathrm{e}^{-\mathrm{i}\phi} - \mathrm{e}^{-\lambda}}{\mathrm{e}^{\lambda} - \mathrm{e}^{-\lambda}}$. and Re stands for the real part. We do not record the expressions of p_b and ρ_b as these are not explicitly required in the remaining part of the paper. The conduction profile $T_\mathrm{b}(z,t)$ contains a linear term in z and a contribution describing damped thermal waves, which propagate into the fluid layer. For low frequencies $T_\mathrm{b}(z,t)$ deviates only slightly from a linear profile. But for high frequency modulation the heat waves enter only into a narrow thermal Stokes boundary layer, thus causing additional exponential spatial behavior.

We now superimpose infinitesimal perturbations on the quiescent basic state and study the stability of the system.

2.2. Linear stability analysis

Let the basic state be perturbed by an infinitesimal thermal perturbation so that

$$\mathbf{q} = \mathbf{q}', \quad p = p_\mathrm{b} + p', \quad T = T_\mathrm{b} + T', \quad \rho = \rho_\mathrm{b} + \rho', \tag{10}$$

where the prime indicates that the quantities are infinitesimal perturbations. Substituting (10) into Eqs. (1)–(4) and using the basic state solutions, we get the linearized equations governing perturbations in the form

$$\frac{\partial \mathbf{q}'}{\partial t} + 2\Omega \times \mathbf{q}' = -\frac{1}{\rho_0}\nabla p' - \beta T' \mathbf{g} + \nu \nabla^2 \mathbf{q}', \tag{11}$$

$$\frac{\partial T'}{\partial t} + w' \frac{\partial T_\mathrm{b}}{\partial z} = \kappa \nabla^2 T'. \tag{12}$$

Since the walls are stress free isothermal, the boundary conditions at $z = 0, d$ are

$$w' = \frac{\partial^2 w'}{\partial z^2} = T' = 0. \tag{13}$$

We render Eqs. (11) and (12) dimensionless by using the transformations

$$(x,y,z) = (x^*, y^*, z^*)d, \quad t = \frac{d^2}{\kappa} t^*,$$
$$(u', v', w') = \frac{\nu}{d}(u^*, v^*, w^*), \tag{14}$$
$$p' = \frac{\mu\kappa}{d^2} p^*, \quad T' = \frac{\nu\kappa}{\beta g d^3} T^*, \quad \omega = \frac{d^2}{\kappa}\bar{\omega},$$

to obtain (after dropping asterisks),

$$\frac{1}{Pr}\frac{\partial \mathbf{q}}{\partial t} + Ta^{1/2}\mathbf{k} \times \mathbf{q} = -\frac{1}{Pr}\nabla p + \frac{1}{Pr} T\mathbf{k} + \nabla^2 \mathbf{q}, \tag{15}$$

$$\left[\frac{\partial}{\partial t} - \nabla^2\right] T = -PrRaw \frac{\partial T_\mathrm{b}}{\partial z}, \tag{16}$$

where $Pr = \nu/\kappa$, $Ta = (2d^2\Omega/\nu)^2$, $Ra = \beta g \Delta T d^3/\nu\kappa$ represent the Prandtl number, Taylor number and Rayleigh number respectively and $\mathbf{k}$ is the unit vector in z-direction. Note that the modulation induced z and t dependence of the vertical conduction temperature profile, implies a (z,t) dependent heat current entering into Eq. (16).

Now to eliminate p from Eq. (15) we operate curl twice on it, which yields an equation for z-component of velocity in the form

$$\left[\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right]\nabla^2 w + Ta^{1/2}\frac{\partial V_z}{\partial z} = \frac{1}{Pr}\nabla_1^2 T, \tag{17}$$

where V_z is z-component of vorticity $\mathbf{V} = \nabla \times \mathbf{q}$, given by the equation

$$\left[\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right] V_z = Ta^{1/2}\frac{\partial w}{\partial z}. \tag{18}$$

In non-dimensional form the boundary conditions (13) become

$$w = \frac{\partial^2 w}{\partial z^2} = T = 0 \quad \text{at } z = 0, 1. \tag{19}$$

After eliminating coupling between Eqs. (16)–(18) we obtain a single equation for vertical component of velocity in the form

$$\left\{\left[\left(\frac{1}{Pr}\frac{\partial}{\partial t}-\nabla^2\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right]\left(\frac{\partial}{\partial t}-\nabla^2\right)\right.$$
$$\left.+Ra\left(\frac{1}{Pr}\frac{\partial}{\partial t}-\nabla^2\right)\frac{\partial T_b}{\partial z}\nabla_1^2\right\}w=0. \tag{20}$$

The boundary conditions (19) can also be expressed in terms of w by making use of Eq. (16), which requires $\partial^4 w/\partial z^4=0$ if w and T are zero. Thus Eq. (20) has to be solved subject to the homogeneous conditions

$$w=\frac{\partial^2 w}{\partial z^2}=\frac{\partial^4 w}{\partial z^4}=\frac{\partial^6 w}{\partial z^6}=0 \quad \text{at } z=0,1. \tag{21}$$

Using Eq. (9), the dimensionless temperature gradient appearing in Eq. (20) may be written as

$$\frac{\partial T_b}{\partial z}=-1+\varepsilon f, \tag{22}$$

where $f=Re[\{A(\lambda)e^{\lambda z}+A(-\lambda)e^{-\lambda z}e^{-i\omega t}]$,

$$A(\lambda)=\frac{\lambda}{2}\left[\frac{e^{-i\phi}-e^{-\lambda}}{e^{\lambda}-e^{-\lambda}}\right] \quad \text{and} \quad \lambda=(1-i)\left(\frac{\omega}{2}\right)^{1/2}.$$

3. Method of solution

We seek the eigenfunctions w and eigenvalues Ra of Eqs. (20) and (21) for a temperature profile given by Eq. (22) that departs from the linear profile $\partial T_b/\partial z=-1$ by quantities of the order ε. It follows that the eigenfunctions and eigenvalues of the present problem differ from those associated with classical Benard problem with rotation by the quantities of order ε. We therefore assume the solution in the form

$$w=w_0+\varepsilon w_1+\varepsilon^2 w_2+\cdots, \tag{23a}$$
$$Ra=Ra_0+\varepsilon^2 Ra_2+\cdots, \tag{23b}$$

where Ra_0 is the Rayleigh number associated with the problem of thermal convection in a rotating fluid layer in the absence of temperature modulation.

Substituting Eq. (23) into Eq. (20) and equating the coefficients of like powers of ε, we obtain the following system of equations:

$$Lw_0=0, \tag{24}$$
$$Lw_1=-Ra_0(L_1)f\nabla_1^2 w_0, \tag{25}$$
$$Lw_2=(L_1)[Ra_2\nabla_1^2 w_0-Ra_0 f\nabla_1^2 w_1], \tag{26}$$

where

$$L\equiv(L_3)(L_2)-Ra_0(L_1)\nabla_1^2$$

with $L_1\equiv\frac{1}{Pr}\frac{\partial}{\partial t}-\nabla^2$, $L_2\equiv\frac{\partial}{\partial t}-\nabla^2$ and $L_3\equiv(L_1)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}$ and w_0, w_1, w_2 are needed to satisfy the boundary conditions (21).

In Eq. (23b) the odd powers of ε are missing, because changing the sign of ε shifts the time origin only which does not affect the problem of stability and thus Ra should be independent of the sign of ε, i.e., Ra_1, $Ra_3, \ldots$ must be zero.

3.1. Solution to the zeroth order problem

The zeroth order problem is equivalent to the rotating Rayleigh–Benard problem in the absence of thermal modulation. The linear analysis of rotating Rayleigh–Benard convection has been thoroughly investigated by Chandrasekhar [12]. We present, below in brief, some of the earlier results in the notations of the present paper for ready reference. The stability of the system in the absence of thermal modulation is investigated by introducing vertical velocity perturbation w_0 as

$$w_0(x,y,z,t)=\exp[\sigma t+i(lx+my)]\sin n\pi z, \tag{27}$$

where σ is the growth rate, l, m are wavenumbers in xy-plane with $l^2+m^2=\alpha^2$. Substituting (27) into Eq. (24) we obtain an expression for Rayleigh number in the form

$$Ra_0=\frac{(\sigma+\alpha^2+n^2\pi^2)[(\alpha^2+n^2\pi^2)(\sigma Pr^{-1}+\alpha^2+n^2\pi^2)^2+n^2\pi^2 Ta]}{\alpha^2(\sigma Pr^{-1}+\alpha^2+n^2\pi^2)}. \tag{28}$$

3.1.1. Stationary convection

For stationary convection σ in Eq. (28) must be real and for marginal stability $\sigma=0$. The corresponding eigenvalue of the Rayleigh number for stationary convection is obtained by putting $\sigma=0$ in Eq. (28), and is given by

$$Ra_0^{(n)}=\frac{1}{\alpha^2}[(\alpha^2+n^2\pi^2)^3+n^2\pi^2 Ta]. \tag{29}$$

For a fixed value of wavenumber α the least eigenvalue occurs for $n=1$, and is given by

$$Ra_0=\frac{1}{\alpha^2}[(\alpha^2+\pi^2)^3+\pi^2 Ta]. \tag{30}$$

Ra_0 assumes the minimum value Ra_{0c} for $\alpha=\alpha_c$, where α_c satisfies the equation

$$(\alpha_c^2)^3+\frac{3\pi^2}{2}(\alpha_c^2)^2-\frac{\pi^2}{2}(Ta+\pi^3)=0. \tag{31}$$

We observe that Eqs. (30) and (31) are the classical results obtained for the problem of thermal instability in a rotating fluid in the absence of modulation (see Chandrasekhar, [12]). Further in the absence of rotation i.e., when $Ta=0$, Eqs. (30) and (31) yield the values $Ra_c=27\pi^4/4=657.5$ and $\alpha_c^2=\pi^2/2$ which are associated with the standard Benard problem (see Chandrasekhar, [12]).

3.1.2. Oscillatory convection

For oscillatory convection σ is in general complex and represented in the form $\sigma=\sigma_r+i\sigma_i$. Substituting $\sigma=i\sigma_i$ and $n=1$ into Eq. (28) one obtain characteristic values of Rayleigh number and the frequency σ_i of the oscillations at the margin of stability in the form,

$$Ra_0^{\mathrm{Osc}} = \frac{2\pi^2}{\alpha^2}\left[\frac{TaPr^2}{(1+Pr)} + (1+Pr)\{3\alpha^4 + \pi^4[1+\alpha^2(3\pi^4+\alpha^4)]\}\right], \quad (32)$$

$$\sigma_i^2 = Pr^2\left[\pi^2\left\{\frac{Ta(1-Pr)}{(\alpha^2+\pi^2)(1+Pr)} - 2\alpha^2\right\} - (\alpha^4+\pi^4)\right]. \quad (33)$$

Further, Eq. (33) yields the necessary condition for the existence of oscillatory motions in the form $Pr < 1$ imposing the condition $\sigma_i^2 > 0$. Thus, it is evident that overstability cannot occur if $Pr \geqslant 1$ and hence under this condition the principle of exchange of stabilities is valid (see Chandrasekhar [12] for details). The case of overstability for low values of Prandtl number is of interest in the presence of temperature modulation because the system then can have a quasi-periodic nature. We study the onset of stationary convection in a rotating fluid layer in the presence of thermal modulation in the present paper. For the simplicity of the notation we denote Ra_0^{St}byRa_0 in the remaining part of the paper.

3.2. Solution to the first order problem

Eq. (25) for w_1 now takes the form

$$Lw_1 = Ra_0\alpha^2(L_1)f\sin\pi z. \quad (34)$$

To simplify this equation, we need

$$(L_1)f\sin\pi z = Re\left[\left\{(\alpha^2+\pi^2) + \mathrm{i}\omega\left(1-\frac{1}{Pr}\right)\right\}f\sin\pi z - 2\pi\lambda f'\cos\pi z\right], \quad (35)$$

where

$$f' = Re[\{A(\lambda)\mathrm{e}^{\lambda z} - A(-\lambda)\mathrm{e}^{-\lambda z}\}\mathrm{e}^{-\mathrm{i}\omega t}].$$

Using Eq. (35) into (34) we obtain

$$Lw_1 = Ra_0\alpha^2 Re[N_1 f\sin\pi z - 2\pi\lambda f'\cos\pi z], \quad (36)$$

where

$$N_1 = (\alpha^2+\pi^2) + \mathrm{i}\omega\left(1-\frac{1}{Pr}\right).$$

We solve Eq. (36) for w_1 by expanding the right-hand side of it in Fourier series expansion and inverting the operator L term by term, to obtain

$$w_1 = Ra_0\alpha^2 Re\left[N_1\sum\left\{\frac{A_n(\lambda)}{L(\omega,n)}\sin n\pi z\mathrm{e}^{-\mathrm{i}\omega t}\right\} - 2\pi\lambda\sum\left\{\frac{B_n(\lambda)}{L(\omega,n)}\cos n\pi z\mathrm{e}^{-\mathrm{i}\omega t}\right\}\right] \quad (37)$$

The detail of algebra is presented in Appendix A through Eqs. (A.1)–(A.9). The equation for w_2, then reads

$$Lw_2 = -Ra_2\alpha^2(\alpha^2+\pi^2)\sin n\pi z + Ra_0\alpha^2 Re[N_2 f w_1 - 2DfDw_1]. \quad (38)$$

We shall not require the solution of this equation but merely use it to determine the Ra_2. The solvability condition requires that the time-independent part of the right-hand side of Eq. (38) must be orthogonal to sin$n\pi z$, and this results in the following equation:

$$Ra_2 = \left[\frac{Ra_0^2\alpha^2}{2(\alpha^2+\pi^2)}\right]Re\left[\sum\frac{N_2^*N_1|A_n(\lambda)|^2L^*(\omega,n)}{|L(\omega,n)|^2} - 4\pi^2|\lambda|^2\sum\frac{nB_n(\lambda)C_n^*(\lambda)L^*(\omega,n)}{|L(\omega,n)|^2}\right]. \quad (39)$$

The detail is presented in Appendix A through Eqs. (A.10)–(A.19). Eq. (38) could now be solved for w_2 if desired, and the procedure may be continued to obtain further corrections to w and Ra. However we shall stop at this step.

The value of Rayleigh number R obtained by this procedure is the eigenvalue corresponding to the eigenfunction w, which, though oscillating, remains bounded in time. Ra is a function of horizontal wavenumber α and the amplitude of modulation ε, accordingly we expand

$$Ra(\alpha,\varepsilon) = Ra_0(\alpha) + \varepsilon^2 Ra_2(\alpha) + \cdots, \quad (40)$$

$$\alpha = \alpha_0 + \varepsilon^2\alpha_2 + \cdots. \quad (41)$$

The critical value of the thermal Rayleigh number Ra is computed up to $O(\varepsilon^2)$ by evaluating Ra_0 and Ra_2 at $\alpha_0 = \alpha_c$ given by Eq. (31). It is only when one wishes to evaluate Ra_4, α_2 must be taken into account [2]. In view of this we write

$$Ra_c(\alpha,\varepsilon) = Ra_{0c}(\alpha_0) + \varepsilon^2 Ra_{2c}(\alpha_0) + \cdots, \quad (42)$$

where Ra_{0c} and Ra_{2c} are respectively given by Eqs. (30) and (39) respectively.

If Ra_{2c} is positive, the effect of modulation is to stabilize the system as compared to the unmodulated system and Ra has minimum at $\varepsilon = 0$. When Ra_{2c} is negative, the effect of modulation is to destabilize the system as compared to the unmodulated system.

To the order of ε^2, Ra_{2c} is obtained for the cases where the oscillating temperature field is (a) symmetric, (b) asymmetric and (c) when only lower wall temperature is oscillating while upper wall is held at constant temperature. The variation of Ra_{2c} with ω for different values of Ta and Pr is depicted in Figs. 1–6 and the results are discussed in the next section.

4. Results and discussion

The expression for the critical correction Rayleigh number Ra_{2c} is computed as a function of the frequency of modulation, Taylor number and Prandtl number and the effect of these parameters on the stability of the system is discussed. The validity of the results obtained here depends on the value of frequency ω of modulation. When ω is small, the period of modulation is large and hence the disturbances will grow to such an extent that the finite amplitude effects become significant. Thus, the assumption of

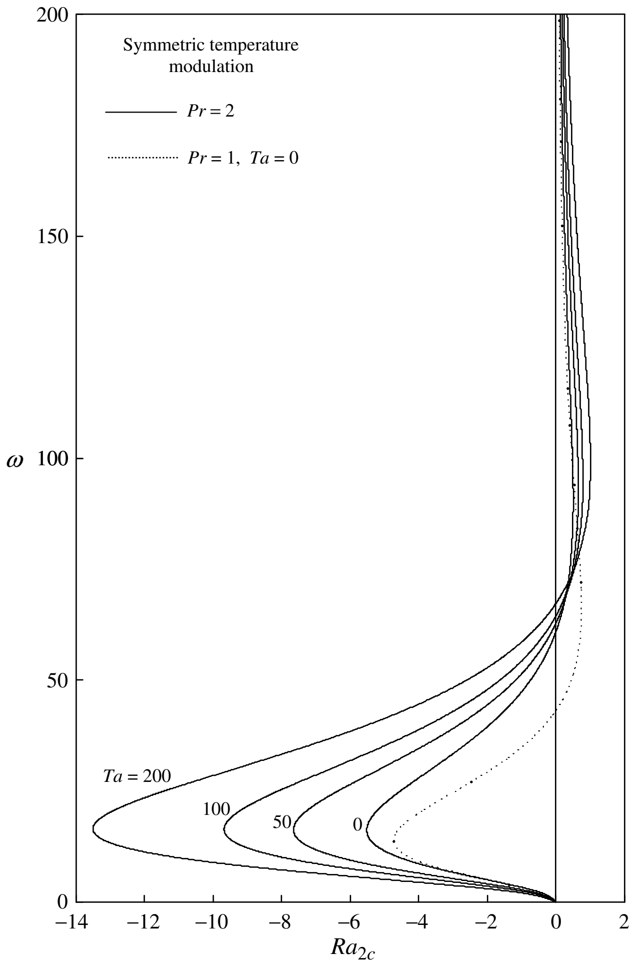

Fig. 1. Variation of Ra_{2c} with ω for different values of Taylor number Ta.

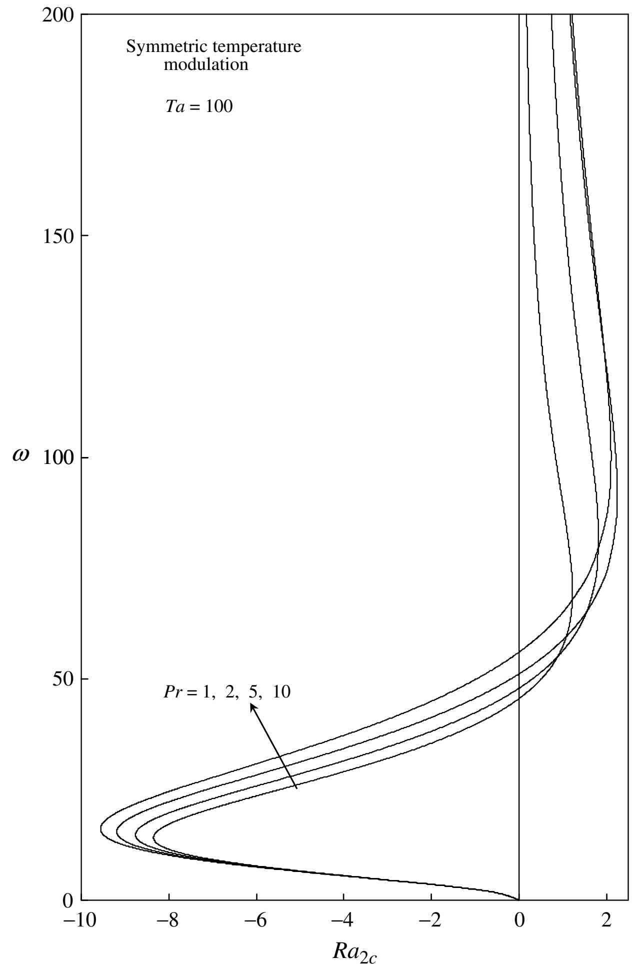

Fig. 2. Variation of Ra_{2c} with ω for different values of Prandtl number Pr.

infinitesimal amplitudes breaks down. On the other hand as $\omega \to \infty$, the effect of thermal modulation is confined only to narrow boundary layer near the boundary and outside this thickness the basic temperature gradient has essentially a uniform gradient. Thus the effect of modulation is significant only for the moderate values of ω.

The variation of the critical correction Rayleigh number Ra_{2c} with the frequency of modulation ω is depicted in Figs. 1 and 2 when the oscillating temperature field is symmetric. It is observed from these figures that for small ω, Ra_{2c} is negative, indicating that the effect of modulation in this case is to destabilize the system; with the convection occurring at lower Rayleigh number than that of a rotating fluid layer in the absence of thermal modulation. This conclusion is consistent with the result of Venezian [2] on the effect of thermal modulation on convection in a non-rotating fluid layer. For moderate and large values ofω, Ra_{2c}becomes positive, indicating that the symmetric modulation has a stabilizing effect for moderate and large values of the frequency. Thus, the symmetric modulation has destabilizing effect on the system for small values of the frequency ω, while it has stabilizing effect for moderate and high values of ω.

Further we observe from these two figures that in each curves there are two peak values of Ra_{2c}, one negative and another positive. The negative peak value is larger than the positive peak value. Let ω^* represent the frequency at which Ra_{2c} changes its sign from negative to positive, then the modulated system may be classified as destabilized or stabilized, compared with the unmodulated system, according as $\omega < \omega^*$ or $\omega > \omega^*$. First Ra_{2c} decreases to its maximum destabilizing value and then increases to its maximum stabilizing value and finally decreases to zero as the frequency increases from zero to infinity. The maximum stabilization or destabilization can be achieved at critical frequencies $\omega = \omega_p$ or $\omega = \omega_n$ depending on the value of the other parameters. Further, at some particular value of the frequency $\omega = \omega_0$, the effect of modulation ceases i.e., $Ra_{2c} = 0$. These critical frequencies depend on the parameter governing the system.

In Fig. 1 the variation of Ra_{2c} with the frequency of modulation ω is shown for different values of Taylor number with Prandtl number kept fixed. We observe from this figure that an increase in the value of Ta increases the magnitude of Ra_{2c}. At small frequencies Ra_{2c} increases negatively, while Ra_{2c} increases positively with the Taylor

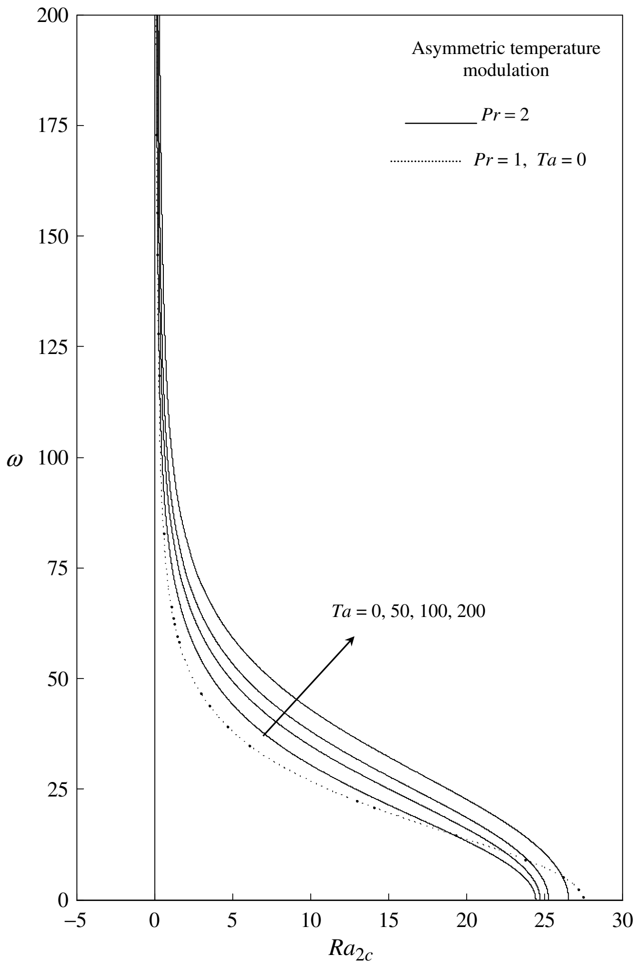

Fig. 3. Variation of Ra_{2c} with ω for different values of Taylor number Ta.

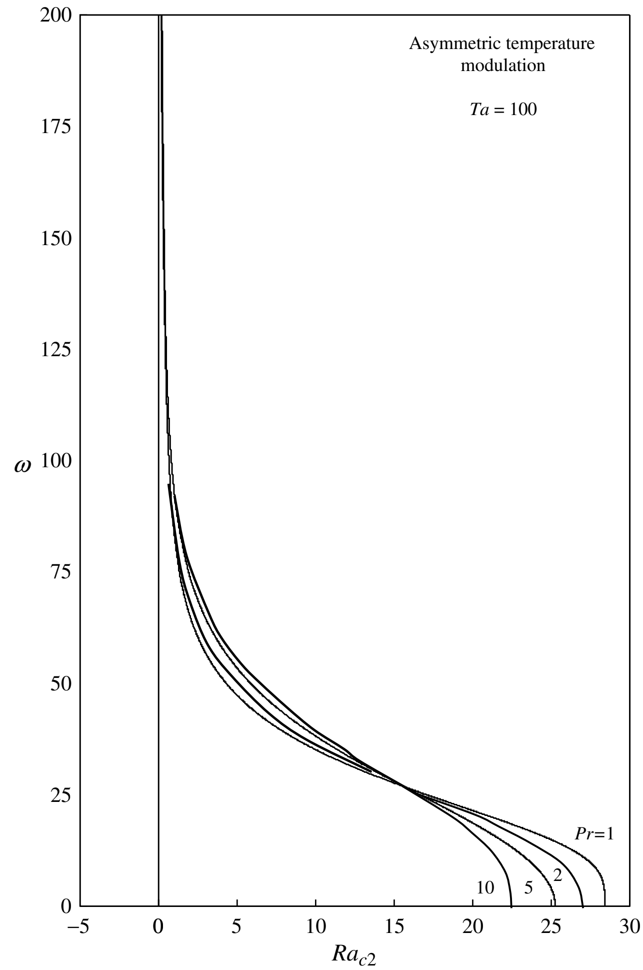

Fig. 4. Variation of Ra_{2c} with ω for different values of Prandtl number Pr.

number at moderate and high frequencies. Thus the effect of rotation is to enhance the instability of the system for small frequencies while its effect is to enhance the stability of the system for moderate and high values of ω. Therefore the rotation reinforces the effect of the symmetric modulation. The dotted curve represents the Venezian [2] result.

The variation of Ra_{2c} with ω for different values of Prandtl number and fixed value of Taylor number is depicted in Fig. 2. This figure indicates that Ra_{2c} increases negatively with Pr for lower value of ω while for higher values ω it increases positively with increasing Pr. Thus, the Prandtl number enhances the instability and stability respectively for low and high frequencies in the case of symmetric modulation. This effect is similar to that of the rotation force. It is interesting to note from Fig. 2 that each curve crosses the other, indicating that for a particular value of the frequency, the critical correction Rayleigh number is same for the corresponding pair of Prandtl number. These results are given in Table 1.

The variation of the critical correction Rayleigh number Ra_{2c} with ω, when the boundary temperature is asymmetric is exhibited through Figs. 3 and 4. It is observed that the system is most stable when frequency is very small. With the increase of ω, the correction Rayleigh number Ra_{2c} decreases. In case of asymmetric modulation, Ra_{2c} is always positive indicating that the convection always sets in at the higher values of Rayleigh number than those predicted for the unmodulated rotating fluid layer. Thus the effect of asymmetric modulation is to delay the onset of convection. This conclusion is consistent with the result of Venezian [2].

The variation of Ra_{2c} with ω for different values Ta and a fixed value of Pr for asymmetric modulation is shown in Fig. 3. The dotted curve represents the Venezian [2] result. This figure indicates that for a given value of modulation frequency, the critical correction Rayleigh number Ra_{2c} increases with the increasing Ta. Indeed, because Ra_{2c} is positive, the applied external force namely, the rotation becomes even more effective in damping out disturbance in a fluid subject to thermal modulation than one subject to an unmodulated system. Therefore the effect of rotation on the asymmetrically modulated fluid layer is to delay the onset of convection. However the effect of rotation disappears for fairly large values of ω, in which case $Ra_{2c} \to 0$. The effect of Prandtl number on Ra_{2c} for asymmetric modulation is indicated in Fig. 4 with Ta kept fixed. We observe

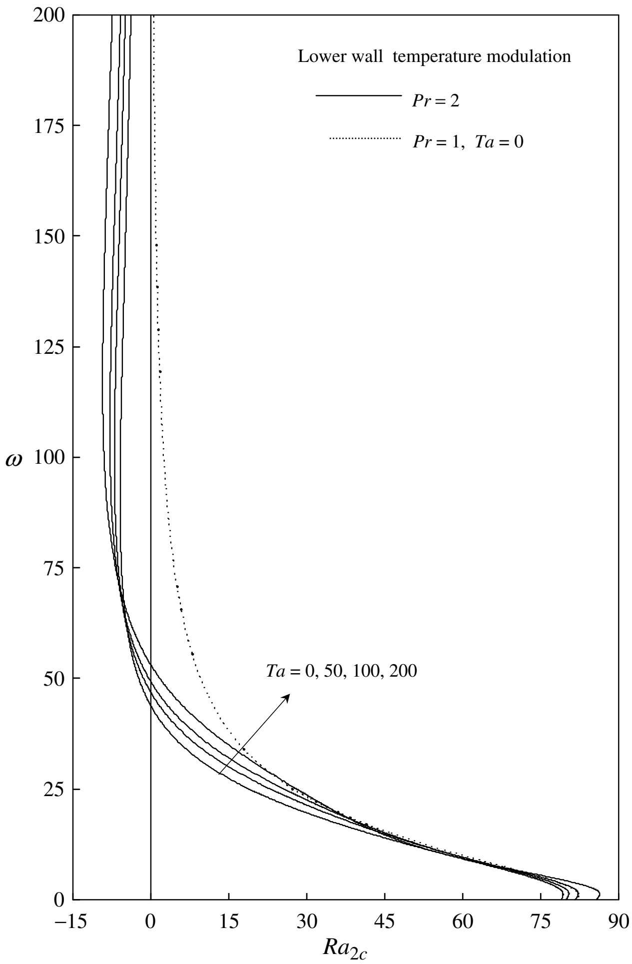

Fig. 5. Variation of Ra_{2c} with ω for different values of Taylor number Ta.

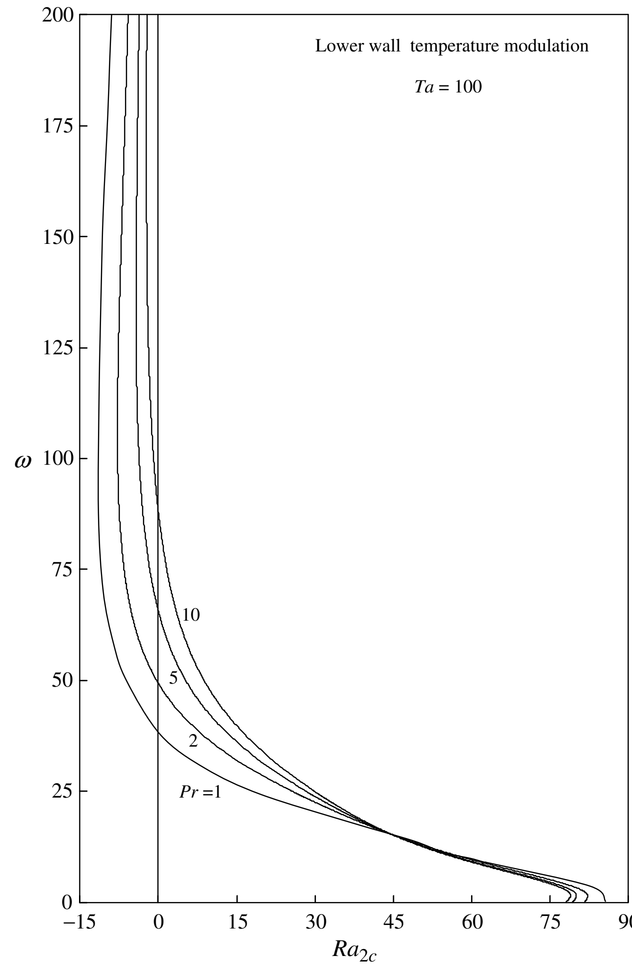

Fig. 6. Variation of Ra_{2c} with ω for different values of Prandtl number Pr.

from this figure that Ra_{2c} decreases with increasing Pr for small values of ω, where as for moderate values of ω, the trend reverses. Therefore Prandtl number reduces the stabilizing effect of asymmetric modulation for small ω, while for moderate and large ω the reverse effect is found.

Variation of Ra_{2c} with ω for lower boundary temperature modulation is shown in Figs. 5 and 6. We find that Ra_{2c} is positive over a range of frequencies, indicating that the lower wall temperature modulation is stabilizing for small to moderate values of the frequency. However, for large frequency, the critical correction Rayleigh number become negative, indicating that the lower wall temperature modulation is destabilizing for such frequencies. Fig. 5 displays the effect of rotation on the stability of the system for lower wall temperature modulation. We observe from this figure that, for low frequencies, Ra_{2c} increases with increasing Ta indicating that the effect of rotation is to delay the onset of convection. Further when ω exceeds certain fixed value, Ra_{2c} becomes negative and the effect of Ta reverses. The dotted curve represents the Venezian [2] result. Fig. 6 reveals the effect of Prandtl number on the correction Rayleigh number for lower wall temperature modulation. It is found that the effect of Pr is similar to that reported in the case of asymmetric modulation.

Table 1
Coincidence of correction Rayleigh number Ra_{2c} for the pair of values of Prandtl number Pr with $Ta = 100$

Pr	ω	Ra_{2c}
(1,2)	55.375	0.861
(1,5)	59.832	1.054
(1,10)	68.001	1.185
(2,5)	65.275	1.525
(2,10)	79.598	1.827
(5,10)	128.007	1.911

5. Conclusions

The effect of thermal modulation on the onset of convection in a horizontal rotating fluid layer is studied using a linear stability analysis and the following conclusions are drawn:

(i) The symmetric modulation destabilizes the system at low frequencies while it stabilizes at moderate and high frequencies.

(ii) The asymmetric modulation is the most stable situation, for all frequencies. The bottom wall temperature modulation has stabilizing effect for small frequencies in the presence of rotation, while it has a destabilizing effect for moderate and large values of the frequency.
(iii) In case of symmetric modulation, rotation destabilizes the system at low frequencies while it stabilizes at moderate and high frequencies. In case of asymmetric modulation rotation always stabilizes the system. In case of lower wall temperature modulation, rotation enhances stability at low frequencies while it enhances instability at large frequencies.
(iv) The effect of Pr on symmetric modulation is similar to that of rotation. The Prandtl number reduces the stabilizing effect of asymmetric modulation for low frequencies while it enhances the stability for moderate frequencies. The Prandtl number has a duel role in case of lower wall temperature modulation.
(v) The effect of both thermal modulation and rotation disappear for large frequency irrespective of the type of thermal modulation.

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Appendix A

We solve Eq. (36) for w_1 by expanding the right-hand side of it in Fourier series expansion and inverting the operator L. For this we need the following Fourier series expansions:

$$e^{\lambda z}\sin m\pi z = \sum_{n=1}^{\infty} g_{nm}(\lambda)\sin n\pi z, \tag{A.1}$$

$$e^{\lambda z}\cos m\pi z = \sum_{n=1}^{\infty} f_{nm}(\lambda)\cos n\pi z, \tag{A.2}$$

where

$$g_{nm}(\lambda) = 2\int_0^1 e^{\lambda z}\sin n\pi z\sin m\pi z\,dz = -\frac{4nm\pi^2\lambda[1+(-1)^{n+m+1}e^{\lambda}]}{[\lambda^2+(n+m)^2\pi^2][\lambda^2+(n-m)^2\pi^2]}, \tag{A.3}$$

$$f_{nm}(\lambda) = 2\int_0^1 e^{\lambda z}\cos n\pi z\cos m\pi z\,dz = -\frac{2\lambda[\lambda^2+(n^2+m^2)\pi^2][1+(-1)^{n+m+1}e^{\lambda}]}{[\lambda^2+(n+m)^2\pi^2][\lambda^2+(n-m)^2\pi^2]}. \tag{A.4}$$

It is convenient to define

$$L(\omega,n) = A_1 + iA_2, \tag{A.5}$$

where

$$A_1 = (\alpha^2+n^2\pi^2)\left[(\alpha^2+\pi^2)^3-(\alpha^2+n^2\pi^2)^3 + \frac{\omega^2}{Pr}(\alpha^2+n^2\pi^2)\left(2+\frac{1}{Pr}\right)+\pi^2 Ta(1-n^2)\right],$$

$$A_2 = \omega\left[(\alpha^2+n^2\pi^2)^3\left(1+\frac{2}{Pr}\right)-\frac{1}{Pr}(\alpha^2+\pi^2)^3 - \frac{\omega^2}{Pr^2}(\alpha^2+n^2\pi^2)+\pi^2 Ta\left(n^2-\frac{1}{Pr}\right)\right]$$

It follows that:

$$L(\sin n\pi z e^{-i\omega t}) = L(\omega,n)\sin n\pi z e^{-i\omega t},\quad L(\cos n\pi z e^{-i\omega t}) = L(\omega,n)\cos n\pi z e^{-i\omega t}. \tag{A.6}$$

Eq. (36) now reads

$$Lw_1 = Ra_0\alpha^2 Re\Big[N_1\sum[\{A(\lambda)g_{n1}(\lambda) + A(-\lambda)g_{n1}(-\lambda)\}\sin n\pi z e^{-i\omega t}] - 2\pi\lambda\sum[\{A(\lambda)f_{n1}(\lambda)-A(-\lambda)f_{n1}(-\lambda)\}\cos nn\pi z e^{-i\omega t}]\Big], \tag{A.7}$$

so that

$$w_1 = Ra_0\alpha^2 Re\left[N_1\sum\left\{\frac{A_n(\lambda)}{L(\omega,n)}\sin n\pi z e^{-i\omega t}\right\} - 2\pi\lambda\sum\left\{\frac{B_n(\lambda)}{L(\omega,n)}\cos n\pi z e^{-i\omega t}\right\}\right], \tag{A.8}$$

where

$$A_n(\lambda) = A(\lambda)g_{n1}(\lambda)+A(-\lambda)g_{n1}(-\lambda),\quad B_n(\lambda) = A(\lambda)f_{n1}(\lambda)-A(-\lambda)f_{n1}(-\lambda). \tag{A.9}$$

To simplify Eq. (26) for w_2, need

$$L_1(fw_1) = Re[N_2 fw_1 - 2DfDw_1], \tag{A.10}$$

where

$$N_2 = (\alpha^2+n^2\pi^2)+i\omega\left(1-\frac{2}{Pr}\right).$$

The equation for w_2, then reads

$$Lw_2 = -Ra_2\alpha^2(\alpha^2+\pi^2)\sin n\pi z + Ra_0\alpha^2 Re[N_2 fw_1 - 2DfDw_1]. \tag{A.11}$$

We shall not require the solution of this equation but merely use it to determine the Ra_2, the first non-zero correction to Ra. The solvability condition requires that the time-independent part of the right-hand side of Eq. (A11) must be orthogonal to $\sin n\pi z$, and therefore,

$$Ra_2 = \left[\frac{2Ra_0}{\alpha^2+\pi^2}\right]Re\left[N_2\int_0^1 \overline{fw_1}\sin\pi z\,dz - 2\int_0^1\overline{DfDw_1}\sin\pi z\,dz\right], \tag{A.12}$$

where the over bar denotes the time average. We have the Fourier series expansions

$$f \sin \pi z = Re \sum A_n(\lambda) \sin n\pi z \mathrm{e}^{-\mathrm{i}\omega t},$$
$$Df \sin \pi z = Re \sum \lambda C_n(\lambda) \sin n\pi z \mathrm{e}^{-\mathrm{i}\omega t}, \tag{A.13}$$

where

$$C_n(\lambda) = A(\lambda) g_{n1}(\lambda) - A(-\lambda) g_{n1}(-\lambda).$$

We also note that the time average of product of two complex functions A and B is given by

$$\overline{A \cdot B} = \frac{1}{2\pi} \int_0^{2\pi} AB \,\mathrm{d}t = \frac{1}{2} A^* B = \frac{1}{2} AB^*, \tag{A.14}$$

where the * denotes a complex conjugate.

Using Eqs. (A13) and (A14) in Eq. (A12) we obtain

$$Ra_2 = \left[\frac{Ra_0^2 \alpha^2}{2(\alpha^2 + \pi^2)}\right] Re \left[\sum \frac{N_2^* N_1 |A_n(\lambda)|^2 L^*(\omega, n)}{|L(\omega, n)|^2} - 4\pi^2 |\lambda|^2 \sum \frac{n B_n(\lambda) C_n^*(\lambda) L^*(\omega, n)}{|L(\omega, n)|^2}\right]. \tag{A.15}$$

We can easily calculate

$$Re[N_2^* N_1 L^*(\omega, n)] = A_3 A_1 + A_4 A_2, \tag{A.16}$$

$$Re[B_n(\lambda) C_n^*(\lambda) L^*(\omega, n)] = \frac{(a_1^2 + b_1^2)}{(d_1 d_2 d_3)^2} [A_1(a_2 a_3 - b_2 b_3) + A_2(a_3 b_2 + a_2 b_3)], \tag{A.17}$$

$$|L(\omega, n)|^2 = A_1^2 + A_2^2, \tag{A.18}$$

$$|A_n(\lambda)|^2 = \frac{4n^2 \pi^4 \omega^2 C_1^2}{d_1 d_2}, \tag{A.19}$$

where

$$A_3 = (\alpha^2 + \pi^2)(\alpha^2 + n^2\pi^2) + \omega^2 \left(1 - \frac{1}{Pr}\right)\left(1 - \frac{2}{Pr}\right),$$
$$A_4 = \omega \left[(\alpha^2 + n^2\pi^2)\left(1 - \frac{1}{Pr}\right) - (\alpha^2 + \pi^2)\left(1 - \frac{2}{Pr}\right)\right],$$
$$a_1 = 4 \sinh\left(\sqrt{\omega/2}\right)\left[C_2 \cosh\left(\sqrt{\omega/2}\right) + C_3 \cos(\sqrt{\omega/2})\right],$$
$$b_1 = 2\left[2C_3 \cosh\left(\sqrt{\omega/2}\right) \sin\left(\sqrt{\omega/2}\right) + C_2 \sin\left(2\sqrt{\omega/2}\right)\right],$$
$$a_2 = \omega^2[2\pi^4(n^2+1)^2 - \{\pi^4(n+1)^2(n-1)^2 - \omega^2\}],$$
$$b_2 = -\omega\pi^2(n^2+1)[2\omega^2 + \{\pi^4(n+1)^2(n-1)^2 - \omega^2\}],$$
$$a_3 = 4n\pi^4\omega^2(n^2+1),$$
$$b_3 = 2n\pi^2\omega[\pi^4(n+1)^2(n-1)^2 - \omega^2],$$
$$d_1 = \omega^2 + \pi^4(n+1)^4, \quad d_2 = \omega^2 + \pi^4(n-1)^4,$$
$$d_3 = 4\left[\sinh^2\left(\sqrt{\omega/2}\right)\cos^2\left(\sqrt{\omega/2}\right) + \cosh^2\left(\sqrt{\omega/2}\right) \times \sin^2\left(\sqrt{\omega/2}\right)\right],$$

and

$$C_1 = 1 + (-1)^{n+2}\mathrm{e}^{-\mathrm{i}\phi}, \quad C_2 = 1 - (-1)^{n+2}\mathrm{e}^{-\mathrm{i}\phi},$$
$$C_3 = (-1)^{n+2} - \mathrm{e}^{-\mathrm{i}\phi}.$$

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